The Solution of the Variable Coefficients Fourth-Order Parabolic Partial Differential Equations by the Homotopy Perturbation Method

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In this work, the homotopy perturbation method proposed by Ji-Huan He [1] is applied to solve both linear and nonlinear boundary value problems for fourth-order partial differential equations. The numerical results obtained with minimum amount of computation are compared with the exact solution to show the efficiency of the method. The results show that the homotopy perturbation method is of high accuracy and efficient for solving the fourth-order parabolic partial differential equation with variable coefficients. The results show also that the introduced method is a powerful tool for solving the fourth-order parabolic partial differential equations.

Key words: Homotopy Perturbation Method; Fourth-Order Parabolic Equation.

1. Introduction

We consider a fourth-order parabolic partial differential equation, with variable coefficients [2, 11, 13, 14]

$$\frac{\partial^2 u}{\partial t^2} + \mu(x,y,z)\frac{\partial^4 u}{\partial x^4} + \lambda(x,y,z)\frac{\partial^4 u}{\partial y^4} + \eta(x,y,z)\frac{\partial^4 u}{\partial z^4} = g(x,y,z,t), \qquad a < x,y,\ z < b, t > 0, \tag{1}$$

where $\mu(x,y,z), \lambda(x,y,z), \eta(x,y,z)$ are variable, subject to the initial conditions [2, 13]

$$u(x,y,z,0) = f_0(x,y,z), \ \ \frac{\partial u}{\partial t}(x,y,z,0) = f_1(x,y,z), \tag{2}$$

and the boundary conditions

$$u(a,y,z,t) = g_0(y,z,t), \ \ u(b,y,z,t) = g_1(y,z,t), \tag{3}$$

$$u(x,a,z,t) = k_0(x,z,t), \ \ u(x,b,z,t) = k_1(x,z,t), \tag{4}$$

$$u(x,y,a,t) = h_0(x,y,t), \ \ u(x,y,b,t) = h_1(x,y,t), \tag{5}$$

$$\begin{aligned} &\frac{\partial^2 u}{\partial x^2}(a,y,z,t) = \overline{g}_0(y,z,t), \\ &\frac{\partial^2 u}{\partial x^2}(b,y,z,t) = \overline{g}_1(y,z,t), \end{aligned} \tag{6}$$

$$\begin{aligned} &\frac{\partial^2 u}{\partial y^2}(x,a,z,t) = \overline{k}_0(x,z,t), \\ &\frac{\partial^2 u}{\partial y^2}(x,b,z,t) = \overline{k}_1(x,z,t), \end{aligned} \tag{7}$$

$$\begin{aligned} &\frac{\partial^2 u}{\partial z^2}(x,y,a,t) = \overline{h}_0(x,y,t), \\ &\frac{\partial^2 u}{\partial z^2}(x,y,b,t) = \overline{h}_1(x,y,t), \end{aligned} \tag{8}$$

where the functions f_i, g_i, k_i, h_i, $\overline{g}_i$, $\overline{k}_i$, $\overline{h}_i$, $i = 0,1$ are continuous. The main focus of researchers was to obtain numerical solutions by using several techniques such as explicit and implicit finite difference schemes used in particular in [3, 4]. In [3] Andrade and McKee studied alternating direction implicit (ADI) methods for fourth-order parabolic equations with variable coefficients. In [5] Byun and Wang investigated fourth-order parabolic equations with weak bounded mean oscillation (BMO) coefficients in Reifenberg domains. Also in [6] Caglar and Caglar investigated fifth-degree B-spline solution for fourth-order parabolic partial differential equations. Conte [4] investigated a stable implicit difference approximation to a fourth-order parabolic equation. Also in [7] Danaee and Evans investigated the fourth-order parabolic equation by using the Hopscotch method. Evans [8] expressed (1) in two space variables as a system of two second-order parabolic equations where finite difference methods were employed. Moreover, in [9] Evans and Yousef investigated the fourth-order parabolic equation with constant coefficients by using the alternating group explicit (AGE) method. Gorman [10] studied fourth-order parabolic partial differential equations in one space variable arises in the transverse vibrations of a uniform flexible beam. Fourth-order parabolic equations of variable coefficients were also studied by Khaliq and Twizell [2] where method of lines (MOL) approach was used to obtain a numerical approximation. In [11] Biazar and Ghazvini investigated the fourth-order parabolic equation with variable coefficients by using the variational iteration method (VIM). Also fourth-order parabolic partial differential equations of constant coefficients were studied by Wazwaz [12] where Adomian's decomposition method (ADM) was used and the noise terms phenomenon were investigated. Wazwaz [13] investigated fourth-order parabolic partial differential equations in higher-dimensional spaces with variable coefficients where Adomian's decomposition method was used to solve them. See also [14] for another research work of this author on fourth-order parabolic partial differential equations. The approach in this paper is different, as we employ a semi-analytic technique which is based on the homotopy perturbation method.

The homotopy perturbation method [1, 15 – 20] is developed to search the accurate asymptotic solutions of nonlinear problems. Also, homotopy perturbation method (HPM) will be effectively used to solve (1). It is well known in the literature that the homotopy perturbation method provides the solution in a rapidly convergent series. This series may provide the solution in a closed form. This technique has been successfully applied to many problems such as functional integral equations [21], Laplace transform [22], quadratic Riccati differential equation [23], hyperbolic partial differential equations [24], integro-differential equations arising in oscillating magnetics fields [25] and parabolic partial differential equations subject to temperature overspecification [26], the second kind of nonlinear integral equations [27], nonlinear equations arising in heat transfer [28], solutions of generalized Hirota-Satsuma coupled KdV equation [29], numerical solutions of the nonlinear Volterra-Fredholm integral equations [30], exact solutions for nonlinear integral equations [31], Fredholm integral equations [32], numeric-analytic solution of system of ODEs [33], nonlinear biochemical reaction model [15], non-linear Fredholm integral equations [34], periodic solutions of nonlinear Jerk equations [35], non-linear system of second-order boundary value problems [36], inverse problem of diffusion equation [37], delay differential equations [38], heat transfer flow of a third grade fluid between parallel plates [39]. Song and Zhang [40] studied application of the extended HPM to a kind of nonlinear evolution equation. Also Yildirim [41] investigated solutions of boundary value problems (BVP) for fourth-order integro-differential equations by HPM. The authors of [42] applied the homotopy perturbation method to solve the Boussinesq partial differential equation arising in modeling of flow in porous media. The homotopy perturbation method is used in [44] to solve the differential algebraic equations. HPM is investigated by authors of [45] to solve the second Painleve equation.

This paper is organized as follows: In Section 2, we describe the homotopy perturbation method briefly and apply this technique to fourth-order parabolic partial differential equations. Section 3 contains several test problems to show the efficiency of the new method. Also a conclusion is given in Section 4. Finally some references are given at the end of this report.

2. Homotopy Perturbation Method

The homotopy perturbation method is a powerful tool for solving various nonlinear equations, especially nonlinear partial differential equations. Recently this method has attracted a wide class of audience in all fields of science and engineering. This method was proposed by the Chinese mathematician J. H. He [1]. In this work, He's homotopy perturbation method is adopted to study the fourth-order parabolic partial differential equations. To illustrate the basic idea of the homotopy perturbation method, consider the following nonlinear equation

$$A(v) = f(r), \quad r \in \Omega, \tag{9}$$

subject to the boundary condition

$$B\left(u, \frac{\partial u}{\partial n}\right) = 0, \quad r \in \Gamma, \tag{10}$$

where A is a general differential operator, B is a boundary operator, $f(r)$ is a known analytic function, Γ is the boundary of the domain Ω and $\frac{\partial}{\partial n}$ denotes differentiation along the normal vector drawn outwards Ω. The operator A can generally be divided into two parts L, N. Therefore (9) can be rewritten as follows:

$$A(v) = L(v) + N(v), \quad r \in \Omega. \tag{11}$$

He [16] constructed a homotopy $v(r, p) : \Omega \times [0, 1] \to R$

which satisfies

$$H(v,p) = (1-p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \tag{12}$$

and is equivalent to

$$H(v,p) = L(v) - L(u_0) + pL(u_0) + p[N(v) - f(r)] = 0, \tag{13}$$

where $p \in [0,1]$ is an embedding operator, and u_0 is an initial approximation of (9). Obviously, we have

$$H(v,0) = L(v) - L(u_0), \quad H(v,1) = A(v) - f(r). \tag{14}$$

The change process of p from zero to unity is just that of $H(v,p)$ from $L(v) - L(u_0)$ to $A(v) - f(r)$. In topology, this is called deformation and $L(v) - L(u_0)$ and $A(v) - f(r)$ are called homotopic. According to the homotopy perturbation method, the parameter p is used as a small parameter, and the solution of (13), can be expressed as in p in the form

$$v = v_0 + v_1 p + v_2 p^2 + \ldots . \tag{15}$$

When $p \to 1$, (13) corresponds to the original one, (9). Thus (15) becomes the approximate solution of (9), i. e.

$$u = \lim_{p\to 1} v = \sum_{k=0}^{\infty} v_k. \tag{16}$$

For solving (1) by the homotopy perturbation method, we have

$$L := \frac{\partial^2}{\partial t^2}, \quad N := \mu(x,y,z)\frac{\partial^4}{\partial x^4} + \lambda(x,y,z)\frac{\partial^4}{\partial y^4} + \eta(x,y,z)\frac{\partial^4}{\partial z^4}, \tag{17}$$

where $g(x,y,z,t)$ is a known function. Beginning with $u_0(x,y,z,t) = f_0(x,y,z) + f_1(x,y,z)t$, the approximate solution of (1) can be determined.

3. Test Problems

To illustrate the solution procedure and show the ability of the method some examples are provided.

Example 1. Consider the following one dimensional variable coefficients fourth-order parabolic partial differential equation [2, 3, 10, 11, 14]

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 u}{\partial x^4}(x,t) = 0, \quad \frac{1}{2} < x < 1, \quad t > 0, \tag{18}$$

subject to the initial conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \tag{19}$$

and the boundary conditions

$$u\left(\frac{1}{2},t\right) = \left(\frac{1+(0.5)^5}{120}\right)\sin(t), \quad u(1,t) = \frac{121}{120}\sin(t), \tag{20}$$

$$\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},t\right) = \frac{1}{6}\left(\frac{1}{2}\right)^3\sin(t), \quad \frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{6}\sin(t).$$

Now by (13), we have:

$$\frac{\partial^2 v}{\partial t^2}(x,t) - \frac{\partial^2 u_0}{\partial t^2}(x,t) + p\frac{\partial^2 u_0}{\partial t^2}(x,t) + p\left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 v}{\partial x^4}(x,t) = 0. \tag{21}$$

From the initial conditions we have

$$u(x,0) = u_0(x,0) + u_1(x,0) + u_2(x,0) + \ldots + u_k(x,0) + \ldots = 0. \tag{22}$$

Using $u_0(x,t) = u(x,0) + tu_t(x,0)$, then we get

$$u_0(x,t) = \left(1 + \frac{x^5}{120}\right)t, \quad u_0(x,0) = 0, \tag{23}$$

and so, we have

$$u_1(x,t) = u_2(x,t) = \ldots|_{t=0} = 0,$$
$$\left.\frac{\partial u}{\partial t}(x,t)\right|_{t=0} = 1 + \frac{x^5}{120} = \tag{24}$$
$$\frac{\partial u_0}{\partial t}(x,0) + \frac{\partial u_1}{\partial t}(x,0) + \ldots + \frac{\partial u_k}{\partial t}(x,0) + \ldots .$$

Thus we can write

$$\frac{\partial u_0}{\partial t}(x,0) = 1 + \frac{x^5}{120}, \quad \frac{\partial u_1}{\partial t}(x,0) = \frac{\partial u_2}{\partial t}(x,0) = \ldots = \frac{\partial u_k}{\partial t}(x,0) = \ldots = 0. \tag{25}$$

Substituting (15) into (21), and equating coefficients of like powers of p, we obtain.

$$p^0 : \frac{\partial^2 v_0}{\partial t^2}(x,t) - \frac{\partial^2 u_0}{\partial t^2}(x,t) = 0,$$

$$p^1: \frac{\partial^2 v_1}{\partial t^2}(x,t) + \frac{\partial^2 u_0}{\partial t^2}(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 v_0}{\partial x^4}(x,t) = 0,$$
$$p^2: \frac{\partial^2 v_2}{\partial t^2}(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 v_1}{\partial x^4}(x,t) = 0,$$
$$\vdots$$
$$p^{k+1}: \frac{\partial^2 v_{k+1}}{\partial t^2}(x,t) + \left(\frac{1}{x} + \frac{x^4}{120}\right)\frac{\partial^4 v_k}{\partial x^4}(x,t) = 0. \quad (26)$$

Therefore, we obtain

$$v_0(x,t) = u_0(x,t) = \left(1 + \frac{x^5}{120}\right)t,$$
$$\frac{\partial^4 v_0}{\partial x^4}(x,t) = xt, \quad (27)$$
$$\frac{\partial^2 v_1}{\partial t^2}(x,t) + 0 + \left(1 + \frac{x^5}{120}\right)t = 0,$$

then we have

$$\frac{\partial v_1}{\partial t}(x,t) = -\left(1 + \frac{x^5}{120}\right)\frac{t^2}{2!} + g(x),$$
$$\left.\frac{\partial u_1}{\partial t}(x,t)\right|_{t=0} = \left.\frac{\partial v_1}{\partial t}(x,t)\right|_{t=0} = g(x). \quad (28)$$

By using (25), (28), we obtain $g(x) = 0$

$$\frac{\partial v_1}{\partial t}(x,t) = -\left(1 + \frac{x^5}{120}\right)\frac{t^2}{2!}, \quad (29)$$

where (29) gives

$$v_1(x,t) = -\left(1 + \frac{x^5}{120}\right)\frac{t^3}{3!}, \quad (30)$$

with repeating this procedure we obtain

$$v_2(x,t) = \left(1 + \frac{x^5}{120}\right)\frac{t^5}{5!},$$
$$\vdots \quad (31)$$
$$v_n(x,t) = (-1)^n\left(1 + \frac{x^5}{120}\right)\frac{t^{2n+1}}{(2n+1)!}.$$

Thus we can write

$$u(x,t) = \lim_{p\to 1}\sum_{k=0}^{\infty} p^k v_k(x,t) = \left(1 + \frac{x^5}{120}\right)\left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \ldots + (-1)^n\frac{t^{2n+1}}{(2n+1)!} + \ldots\right) = \left(1 + \frac{x^5}{120}\right)\sin(t), \quad (32)$$

which is the exact solution of the test example 1.

Example 2. Consider the following parabolic equation [10, 11, 14]

$$\frac{\partial^2 u}{\partial t^2}(x,t) + \left(\frac{x}{\sin(x)} - 1\right)\frac{\partial^4 u}{\partial x^4}(x,t) = 0, \quad 0 < x < 1, \quad t > 0, \quad (33)$$

with initial conditions

$$u(x,0) = (x - \sin(x)), \quad \frac{\partial u}{\partial t}(x,0) = -(x - \sin(x)), \quad (34)$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \exp(-t)(1 - \sin(1)),$$
$$\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1,t) = \exp(-t)\sin(1). \quad (35)$$

From the initial conditions we have

$$u(x,t) = \sum_{k=0}^{\infty} u_k(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots$$
$$x - \sin(x) = u(x,0) = u_0(x,0) + u_1(x,0) + u_2(x,0) + \ldots . \quad (36)$$

Using $u_0(x,t) = u(x,0) + t u_t(x,0)$, we obtain

$$u_0(x,t) = (x - \sin(x))(1 - t),$$
$$u_1(x,0) = u_2(x,0) = \ldots = 0,$$
$$\frac{\partial u}{\partial t}(x,0) = \frac{\partial u_0}{\partial t}(x,0) + \frac{\partial u_1}{\partial t}(x,0) + \frac{\partial u_2}{\partial t}(x,0) + \ldots$$
$$-x + \sin(x) = \frac{\partial u}{\partial t}(x,0) = \frac{\partial u_0}{\partial t}(x,0) + \frac{\partial u_1}{\partial t}(x,0) + \frac{\partial u_2}{\partial t}(x,0) + \ldots . \quad (37)$$

Thus we obtain

$$\frac{\partial u_0}{\partial t}(x,0) = -(x - \sin(x)),$$
$$\frac{\partial u_1}{\partial t}(x,0) = \frac{\partial u_2}{\partial t}(x,0) = \ldots = 0. \quad (38)$$

Commencing with $u_0(x,t) = (x - \sin(x))(1 - t)$, and with equating coefficients of like powers of p, we obtain

$$p^0: \frac{\partial^2 v_0}{\partial t^2}(x,t) - \frac{\partial^2 u_0}{\partial t^2}(x,t) = 0,$$
$$p^1: \frac{\partial^2 v_1}{\partial t^2}(x,t) + \frac{\partial^2 u_0}{\partial t^2}(x,t) + \left(\frac{x}{\sin(x)} - 1\right)\frac{\partial^4 v_0}{\partial x^4}(x,t) = 0,$$

$$p^2 : \frac{\partial^2 v_2}{\partial t^2}(x,t) + \left(\frac{x}{\sin(x)} - 1\right)\frac{\partial^4 v_1}{\partial x^4}(x,t) = 0,$$
$$\vdots$$
$$p^{k+1} : \frac{\partial^2 v_{k+1}}{\partial t^2}(x,t) + \left(\frac{x}{\sin(x)} - 1\right)\frac{\partial^4 v_k}{\partial x^4}(x,t) = 0. \quad (39)$$

This gives

$$\begin{aligned} v_0(x,t) &= u_0(x,t) = (x - \sin(x))(1-t), \\ v_1(x,t) &= (x-\sin(x))\left(\frac{t^2}{2!} - \frac{t^3}{3!}\right), \\ v_2(x,t) &= (x-\sin(x))\left(\frac{t^4}{4!} - \frac{t^5}{5!}\right), \\ &\vdots \end{aligned} \quad (40)$$

Using (40) yields

$$\begin{aligned} u(x,t) &= \lim_{p\to 1}\sum_{k=0}^{\infty} p^k v_k(x,t) = \sum_{k=0}^{\infty} v_k(x,t), \\ u(x,t) &= (x-\sin(x))\left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right) \\ &= \exp(-t)(x - \sin(x)), \end{aligned} \quad (41)$$

which is the exact solution of the test example 2.

Example 3. Now we solve the following one dimensional non-homogeneous fourth-order parabolic equation [10, 11, 14]

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(x,t) + (1+x)\frac{\partial^4 u}{\partial x^4}(x,t) = \\ &\quad \left(x^3 + x^4 - \left(\frac{6}{7!}\right)x^7\right)\cos(t), \\ &\quad 0 < x < 1, \quad t > 0, \end{aligned} \quad (42)$$

subject to the initial conditions

$$u(x,0) = \left(\frac{6}{7!}\right)x^7, \quad \frac{\partial u}{\partial t}(x,0) = 0, \quad (43)$$

and the boundary conditions

$$\begin{aligned} &u(0,t) = 0, \quad u(1,t) = \left(\frac{6}{7!}\right)\cos(t), \\ &\frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad \frac{\partial^2 u}{\partial x^2}(1,t) = \frac{1}{20}\cos(t). \end{aligned} \quad (44)$$

Starting with $u_0(x,t) = (\frac{6}{7!})x^7$, and equating coefficients of like powers of p, we obtain

$$\begin{aligned} p^0 &: v_0(x,t) = \left(\frac{6}{7!}\right)x^7, \\ p^1 &: v_1(x,t) = (x^3+x^4)\left(1 - \frac{t^2}{2!} - \cos(t)\right) \\ &\quad - \left(\frac{6}{7!}\right)x^7(1-\cos(t)), \\ p^2 &: v_2(x,t) = 24(1+x)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cos(t)\right) \\ &\quad - (x^3+x^4)\left(1 - \frac{t^2}{2!} - \cos(t)\right), \\ p^3 &: v_3(x,t) = -24(1+x)\left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \cos(t)\right), \end{aligned} \quad (45)$$

$$\begin{aligned} &v_i(x,t) = 0, \quad \forall\, i = 4,5,6,\ldots, \\ &v(x,t) = \sum_{k=0}^{\infty} v_k(x,t), \end{aligned} \quad (46)$$

where

$$\begin{aligned} u(x,t) &= \sum_{k=0}^{\infty} u_k(x,t) = \lim_{p\to 1}\sum_{k=0}^{\infty} p^k v_k(x,t) \\ &= \frac{6}{7!}x^7\cos(t), \end{aligned} \quad (47)$$

which is the exact solution of the test example 3.

Example 4. Consider the fourth-order parabolic equation in two space variables [2, 3, 11, 13]

$$\begin{aligned} &\frac{\partial^2 u}{\partial t^2}(x,y,t) + 2\left(\frac{1}{x^2} + \frac{x^4}{6!}\right)\frac{\partial^4 u}{\partial x^4}(x,y,t) \\ &+ 2\left(\frac{1}{y^2} + \frac{y^4}{6!}\right)\frac{\partial^4 u}{\partial y^4}(x,y,t) = 0, \\ &\frac{1}{2} < x,y < 1, \quad t > 0, \end{aligned} \quad (48)$$

with initial conditions

$$u(x,y,0) = 0, \quad \frac{\partial u}{\partial t}(x,y,0) = 2 + \frac{x^6}{6!} + \frac{y^6}{6!}, \quad (49)$$

and the boundary conditions [2, 13]

$$\begin{aligned} &u\left(\frac{1}{2},y,t\right) = \left(2 + \frac{(0.5)}{6!} + \frac{y^6}{6!}\right)\sin(t), \\ &u(1,y,t) = \left(2 + \frac{1}{6!} + \frac{y^6}{6!}\right)\sin(t), \\ &\frac{\partial^2 u}{\partial x^2}\left(\frac{1}{2},y,t\right) = \frac{(0.5)^4}{24}\sin(t), \ \frac{\partial^2 u}{\partial x^2}(1,y,t) = \frac{1}{24}\sin(t), \\ &\frac{\partial^2 u}{\partial y^2}\left(x,\frac{1}{2},t\right) = \frac{(0.5)^4}{24}\sin(t), \ \frac{\partial^2 u}{\partial y^2}(x,1,t) = \frac{1}{24}\sin(t). \end{aligned} \quad (50)$$

Commencing with $u_0(x,y,t) = (2+\frac{x^6}{6!}+\frac{y^6}{6!})t$, and equating coefficients of like powers of p, we obtain

$$\begin{aligned}
p^0 &: \frac{\partial^2 v_0}{\partial t^2}(x,y,t) - \frac{\partial^2 u_0}{\partial t^2}(x,y,t) = 0,\\
p^1 &: \frac{\partial^2 v_1}{\partial t^2}(x,y,t) + \frac{\partial^2 u_0}{\partial t^2}(x,y,t)\\
&\quad + 2\left(\frac{1}{x}+\frac{x^4}{6!}\right)\frac{\partial^4 v_0}{\partial x^4}(x,y,t)\\
&\quad + 2\left(\frac{1}{y}+\frac{y^4}{6!}\right)\frac{\partial^4 v_0}{\partial y^4}(x,y,t) = 0, \qquad (51)\\
p^2 &: \frac{\partial^2 v_2}{\partial t^2}(x,y,t) + 2\left(\frac{1}{x}+\frac{x^4}{6!}\right)\frac{\partial^4 v_1}{\partial x^4}(x,y,t)\\
&\quad + 2\left(\frac{1}{y}+\frac{y^4}{6!}\right)\frac{\partial^4 v_1}{\partial y^4}(x,y,t) = 0,\\
&\vdots
\end{aligned}$$

Thus we obtain

$$\begin{aligned}
v_0(x,y,t) &= \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)t,\\
v_1(x,y,t) &= -\left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^3}{3!}, \qquad (52)\\
v_2(x,y,t) &= \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\frac{t^5}{5!},\\
&\vdots
\end{aligned}$$

It can be seen that

$$\begin{aligned}
u(x,y,t) &= \lim_{p\to 1}\sum_{k=0}^{\infty} p^k v_k(x,y,t)\\
&= \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\left(t-\frac{t^3}{3!}+\frac{t^5}{5!}-\frac{t^7}{7!}+\dots\right), \qquad (53)\\
u(x,y,t) &= \left(2+\frac{x^6}{6!}+\frac{y^6}{6!}\right)\sin(t),
\end{aligned}$$

which is the exact solution of the test example 4.

Example 5. Suppose we solve the following partial differential equation in three space variables [11,13]

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(x,y,z,t) + \left(\frac{y+z}{2\cos(x)}-1\right)\frac{\partial^4 u}{\partial x^4}(x,y,z,t)\\
&+ \left(\frac{x+z}{2\cos(y)}-1\right)\frac{\partial^4 u}{\partial y^4}(x,y,z,t)\\
&+ \left(\frac{y+x}{2\cos(z)}-1\right)\frac{\partial^4 u}{\partial z^4}(x,y,z,t) = 0,\\
&0 < x,y,z < \frac{\pi}{3}, \quad t>0, \qquad (54)
\end{aligned}$$

subject to the initial conditions

$$u(x,y,z,0) = x+y+z-(\cos(x)+\cos(y)+\cos(z)), \quad (55)$$

$$\begin{aligned}
\frac{\partial u}{\partial t}(x,y,z,0) &= (\cos(x)+\cos(y)+\cos(z))\\
&\quad -(x+y+z), \qquad (56)
\end{aligned}$$

and the boundary conditions [13]

$$\begin{aligned}
&u(0,y,z,t) = \exp(-t)(-1+y+z-\cos(y)-\cos(z)),\\
&u(\frac{\pi}{3},y,z,t) =\\
&\exp(-t)\left(\frac{2\pi-3}{6}+y+z-\cos(y)-\cos(z)\right),\\
&u(x,0,z,t) =\\
&\exp(-t)(-1+x+z-\cos(x)-\cos(z)),\\
&u\left(x,\frac{\pi}{3},z,t\right) =\\
&\exp(-t)\left(\frac{2\pi-3}{6}+x+z-\cos(x)-\cos(z)\right),\\
&u(x,y,0,t) = \qquad (57)\\
&\exp(-t)(-1+x+y-\cos(x)-\cos(y)),\\
&u\left(x,y,\frac{\pi}{3},t\right) =\\
&\exp(-t)\left(\frac{2\pi-3}{6}+x+y-\cos(x)-\cos(y)\right),\\
&\frac{\partial u}{\partial x}(0,y,z,t) = \frac{\partial u}{\partial y}(x,0,z,t) = \frac{\partial u}{\partial z}(x,y,0,t) = \exp(-t),\\
&\frac{\partial u}{\partial x}\left(\frac{\pi}{3},y,z,t\right) = \frac{\partial u}{\partial y}\left(x,\frac{\pi}{3},z,t\right)\\
&= \frac{\partial u}{\partial z}\left(x,y,\frac{\pi}{3},t\right) = \left(\frac{\sqrt{3}+2}{2}\right)\exp(-t).
\end{aligned}$$

From (13) the homotopy perturbation method will be obtained as

$$\frac{\partial^2 v}{\partial t^2}(x,y,z,t) - \frac{\partial^2 u_0}{\partial t^2}(x,y,z,t) + p\frac{\partial^2 u_0}{\partial t^2}(x,y,z,t)$$

$$+p\left[\left(\frac{y+z}{2\cos(x)}-1\right)\frac{\partial^4 v}{\partial x^4}(x,y,z,t)+\left(\frac{x+z}{2\cos(y)}-1\right)\frac{\partial^4 v}{\partial y^4}(x,y,z,t)+\left(\frac{y+x}{2\cos(z)}-1\right)\frac{\partial^4 v}{\partial z^4}(x,y,z,t)\right]=0. \quad (58)$$

Starting with $u_0(x,y,z,t) = (x + y + z - \cos(x) - \cos(y) - \cos(z))(1 - t)$, by using (15) for (58) and equating coefficients of like powers of p yield

$$\begin{aligned}
p^0 &: \frac{\partial^2 v_0}{\partial t^2}(x,y,z,t)-\frac{\partial^2 u_0}{\partial t^2}(x,y,z,t)=0,\\
p^1 &: \frac{\partial^2 v_1}{\partial t^2}(x,y,z,t)+\frac{\partial^2 u_0}{\partial t^2}(x,y,z,t)\\
&+\left(\frac{y+z}{2\cos(x)}-1\right)\frac{\partial^4 v_0}{\partial x^4}(x,t)\\
&+\left(\frac{x+z}{2\cos(y)}-1\right)\frac{\partial^4 v_0}{\partial y^4}(x,y,z,t)\\
&+\left(\frac{y+x}{2\cos(z)}-1\right)\frac{\partial^4 v_0}{\partial z^4}(x,y,z,t)=0,\\
&\vdots
\end{aligned} \quad (59)$$

Therefore, we have

$$\begin{aligned}
&v_0(x,y,z,t)=\\
&(x+y+z-\cos(x)-\cos(y)-\cos(z))(1-t),\\
&v_1(x,y,z,t)=\\
&(x+y+z-\cos(x)-\cos(y)-\cos(z))\left(\frac{t^2}{2!}-\frac{t^3}{3!}\right),\\
&v_2(x,y,z,t)=\\
&(x+y+z-\cos(x)-\cos(y)-\cos(z))\left(\frac{t^4}{4!}-\frac{t^5}{5!}\right),\\
&\vdots
\end{aligned} \quad (60)$$

Thus we can write

$$\begin{aligned}
&u(x,y,z,t)=\lim_{p\to 1}\sum_{k=0}^{\infty}p^k v_k(x,,y,z,t)=\\
&(x+y+z-\cos(x)-\cos(y)-\cos(z))\left(1-t+\frac{t^2}{2!}\right.\\
&\left.-\frac{t^3}{3!}+\frac{t^4}{4!}+\dots\right)=(x+y+z-\cos(x)-\cos(y)\\
&-\cos(z))\sum_{k=0}^{\infty}\frac{(-t)^n}{n!}=(x+y+z-\cos(x)-\cos(y)\\
&-\cos(z))\exp(-t),
\end{aligned} \quad (61)$$

which is the exact solution of the test example 5.

Example 6. As the last example, we consider the following three dimensional non-homogeneous fourth-order parabolic equation [11, 13]

$$\begin{aligned}
&\frac{\partial^2 u}{\partial t^2}(x,y,z,t)+\frac{1}{4!}\left[\frac{1}{z}\frac{\partial^4 u}{\partial x^4}(x,y,z,t)\right.\\
&\left.+\frac{1}{x}\frac{\partial^4 u}{\partial y^4}(x,y,z,t)+\frac{1}{y}\frac{\partial^4 u}{\partial z^4}(x,y,z,t)\right]=\\
&-\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}+\frac{1}{x^5}+\frac{1}{y^5}+\frac{1}{z^5}\right)\cos(t),\\
&\frac{1}{2}<x,y,z<1,\quad t>0,
\end{aligned} \quad (62)$$

with initial conditions

$$u(x,y,z,0)=\frac{x}{y}+\frac{y}{z}+\frac{z}{x},\quad \frac{\partial u}{\partial t}(x,y,z,0)=0, \quad (63)$$

and the boundary conditions [13]

$$\begin{aligned}
u\left(\frac{1}{2},y,z,t\right)&=\left(\frac{1}{2y}+\frac{y}{z}+2z\right)\cos(t),\\
u(1,y,z,t)&=\left(\frac{1}{y}+\frac{y}{z}+z\right)\cos(t),\\
u\left(x,\frac{1}{2},z,t\right)&=\left(2x+\frac{1}{2z}+\frac{z}{x}\right)\cos(t),\\
u(x,1,z,t)&=\left(x+\frac{1}{z}+\frac{z}{x}\right)\cos(t),\\
u\left(x,y,\frac{1}{2},t\right)&=\left(\frac{x}{y}+2y+\frac{1}{2x}\right)\cos(t),\\
u(x,y,1,t)&=\left(\frac{x}{y}+y+\frac{1}{x}\right)\cos(t),\\
\frac{\partial u}{\partial x}\left(\frac{1}{2},y,z,t\right)&=\left(\frac{1}{y}-4z\right)\cos(t),\\
\frac{\partial u}{\partial x}(1,y,z,t)&=\left(\frac{1}{y}-z\right)\cos(t),\\
\frac{\partial u}{\partial y}\left(x,\frac{1}{2},z,t\right)&=\left(-4x+\frac{1}{z}\right)\cos(t),\\
\frac{\partial u}{\partial y}(x,1,z,t)&=\left(-x+\frac{1}{z}\right)\cos(t),\\
\frac{\partial u}{\partial z}\left(x,y,\frac{1}{2},t\right)&=\left(-4y+\frac{1}{x}\right)\cos(t),\\
\frac{\partial u}{\partial z}(x,y,1,t)&=\left(-y+\frac{1}{x}\right)\cos(t).
\end{aligned} \quad (64)$$

Applying the He's homotopy perturbation method and

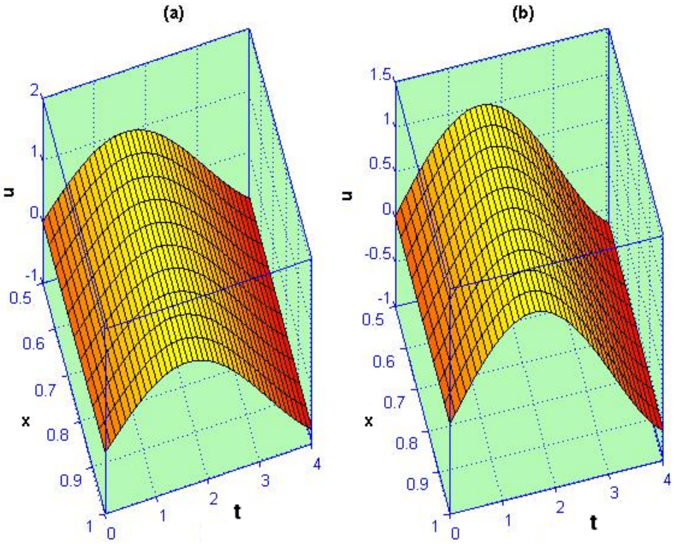

Fig. 1. The surface of the exact and the approximate solutions of (18). (a) The exact solution. (b) The approximate solution obtained in this work.

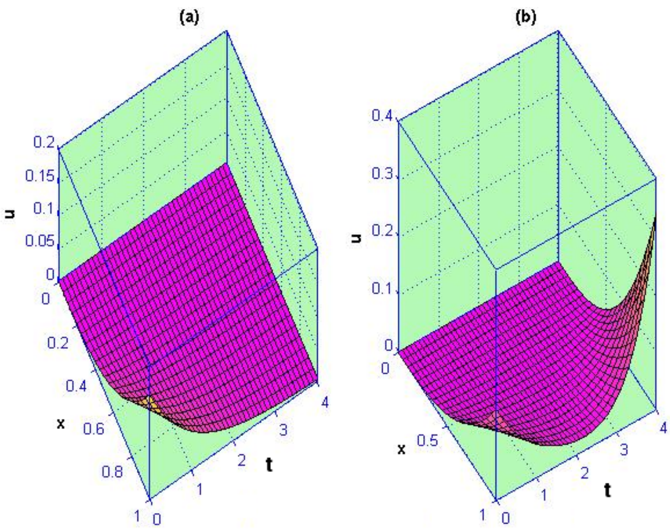

Fig. 2. The surface of the exact and the approximate solutions of (33). (a) The exact solution. (b) The approximate solution obtained in this work.

equating coefficients of like power of p, we obtain

$$\begin{aligned}
p^0 &: v_0(x,y,z,t) = \frac{x}{y} + \frac{y}{z} + \frac{z}{x}, \\
p^1 &: v_1(x,y,z,t) = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)(\cos(t) - 1) \\
&\quad + \left(\frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5}\right)\left(\cos(t) - \frac{t^2}{2!} - 1\right), \\
p^2 &: v_2(x,y,z,t) = \left(\frac{1}{x^5} + \frac{1}{y^5} + \frac{1}{z^5}\right)\left(\cos(t) + \frac{t^2}{2!} - 1\right) \\
&\quad + 70\left(\frac{1}{zx^9} + \frac{1}{xy^9} + \frac{1}{yz^9}\right)\left(\frac{t^2}{2!} + \frac{t^4}{4!} + \cos(t) - 1\right), \\
&\vdots
\end{aligned} \tag{65}$$

Thus we can write

$$u(x,y,z,t) = \left(\frac{x}{y} + \frac{y}{z} + \frac{z}{x}\right)\cos(t), \tag{66}$$

which is the exact solution of the test example 6.

We illustrate the accuracy and efficiency of homotopy perturbation method (HPM) by applying the method to fourth-order parabolic equations and comparing the approximate solutions with the exact solutions. For this purpose, we calculate the numerical results of the exact solutions and the multi-terms approximate solutions of HPM. At the same time, the surface graphics of the exact and multi-terms approximate solutions are plotted in Figs. 1, 2, 3, 4, 5 and 6. One can

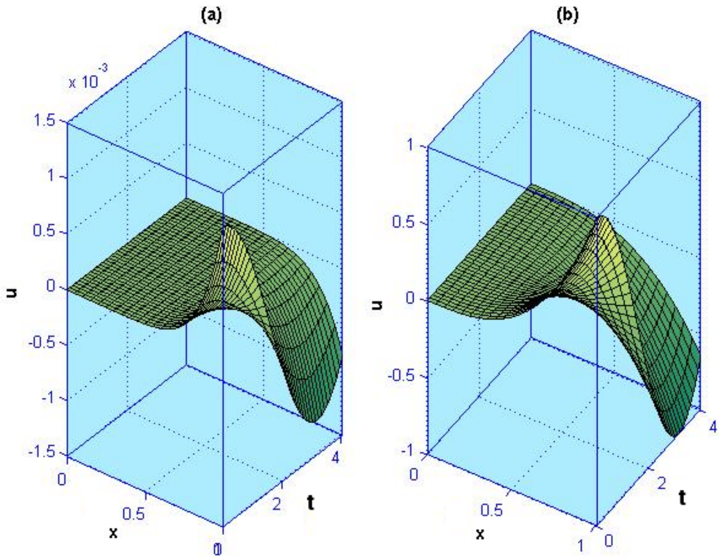

Fig. 3. The surface of the exact and the approximate solutions of (42). (a) The exact solution. (b) The approximate solution obtained in this work.

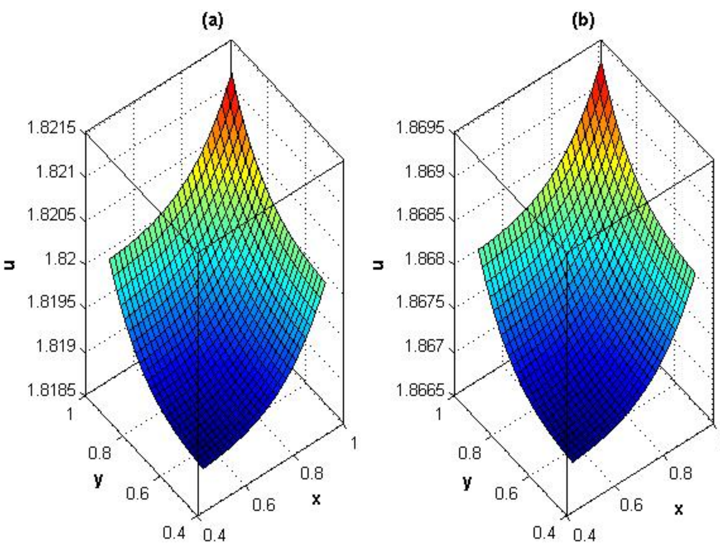

Fig. 4. The surface of the exact and the approximate solutions of (48). (a) The exact solution. (b) The approximate solution obtained in this work.

see that the approximate solutions obtained by HPM are quite close to their exact solutions.

4. Conclusion

The main idea of this work was to propose a simple method for solving fourth-order parabolic partial differential equations. We have achieved an analytical solution by applying the He's homotopy perturbation method (HPM). The main advantage of the method is the fact that it gives an analytical approximation solution. The results are compared with those in the literature, revealing that the obtained solutions are exactly the same with those obtained by the Adomian's decomposition method [12 – 14]. Also solutions obtained by the homotopy perturbation method are the same with He's variation iteration method [11]. In examples we observed that the HPM with the initial approximations obtained from (16) yield exact solutions in few iterations only. In all examples we observed that the HPM solutions are more efficient than the modified Adomian's decomposition method. HPM avoids the difficulties arising in finding the Adomian's polynomials [46 – 50]. In addition, the calculations involved in HPM are very simple and straightforward. It can be shown that the HPM is a promising tool for solv-

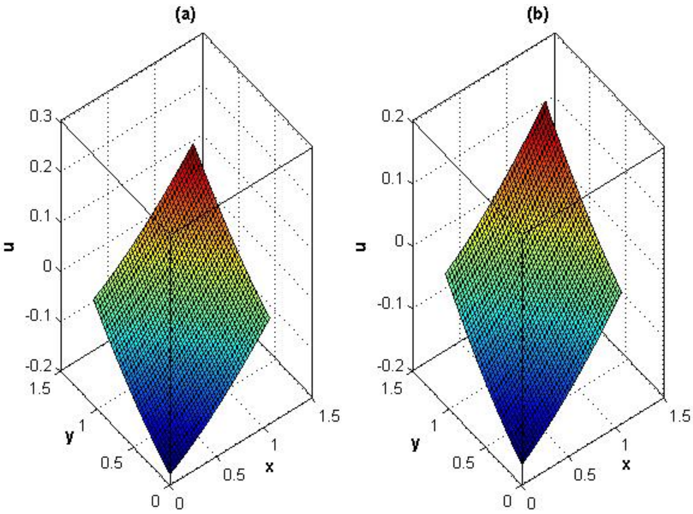

Fig. 5. The surface of the exact and the approximate solutions of (54). (a) The exact solution. (b) The approximate solution obtained in this work.

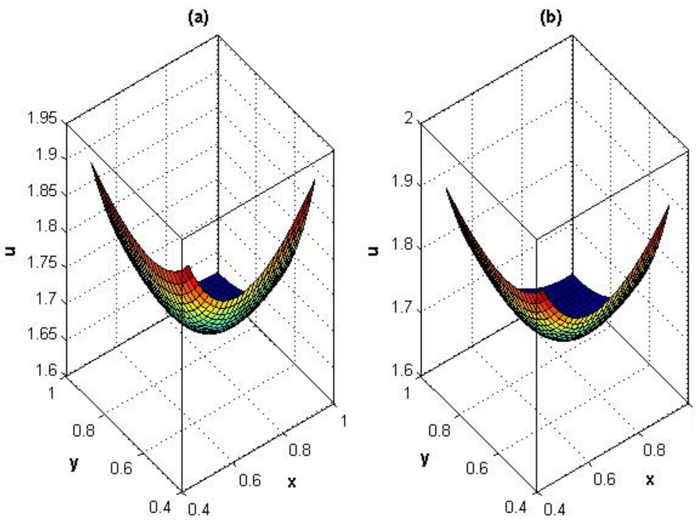

Fig. 6. The surface of the exact and the approximate solutions of (63). (a) The exact solution. (b) The approximate solution obtained in this work.

ing some linear and nonlinear partial differential equations. It is worth to point out that this technique unlike the mesh points methods [43] does not provide any linear or nonlinear system of equations.

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